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A Theorem on Linear Diophantine Equations and the Paging-Complexity of Loop-Chains

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Abstract — Zusammenfassung

A Theorem on Linear Diophantine Equations and the Paging-Complexity of Loop-Chains. For a given reference-structure the following problem is posed: Find a partition of a program's (resp. its data) addresses in pages of limited size and a paging-algorithm such that the "page-fault-rate" is minimized. A "paging-complexity" (of reference-structures) is introduced in order to measure optimal performance. It is shown that the paging-complexity of a concatenation of simple loops is the arithmetic mean of the paging-complexities of the loops involved. The proof rests on a theorem on linear diophantine equations which may deserve attention for its own sake.

Ein Satz über lineare diophantische Gleichungen und die Seitenwechselkomplexität von Schleifenketten. Für eine gegebene Referenzenstruktur stellt sich folgendes Problem: Finde eine Partition der Programm- (bzw. Daten-) Adressen in Seiten beschränkter Größe und einen Seitenwechselalgorithmus derart, daß die „Fehlseitenrate" minimiert wird. Das Besterreichbare wird durch die „Seitenwechselkomplexität" gemessen. Es wird gezeigt, daß die Seitenwechselkomplexität einer Konkatenation einfacher Schleifen das arithmetische Mittel der Seitenwechselkomplexitäten der beteiligten Schleifen ist. Der Beweis stützt sich auf einen Satz über lineare diophantische Gleichungen, der für sich Beachtung verdienen mag.

1. Introduction and Definitions

The problem of managing paged memory systems has been intensively studied in the past ([1]). Among the formal approaches the following are prominent:

(i) Based on stochastic assumptions on the page-reference behaviour of programs various paging-algorithms (i.e. algorithms which govern the exchange of pages between main memory and auxiliary storage devices) are analyzed; in some sense optimal algorithms can be obtained for subclasses of reference behaviour.

(ii) Code and data of programs are partitioned in blocks (pages) in order to achieve a "low rate of interreference" between blocks, thus obviating the need for frequently swapping information between the directly accessible storage media and the auxiliary devices.

Both, (i) and (ii) have been considered inadequate for different reasons (cf. [3]). An alternative approach, in a way combining (i) and (ii), has been suggested in [5] and [6]: firstly, the *address*-reference behaviour of single programs is modelled deterministically by means of reference-structures. Then, given a particular reference-structure, the following problem is posed: Find a partition of the program's addresses in pages of limited size (a "pagination") and a paging-algorithm such that the "page-fault rate" (i.e. the relative frequency of not finding a referenced page in main memory) is minimized. Both, pagination and paging-algorithm may be restricted to limited classes. In [6] this problem has been solved for a reference-structure called "simple loop" with respect to arbitrary paginations and demand-paging-algorithms (cf. [1]). The purpose of this note is to show that the "paging-complexity" (loosely speaking, this is the minimal page-fault rate) of a concatenation of simple loops (a "loop-chain") is just the arithmetic mean of the paging-complexities of the simple loops involved. The proof employs a theorem on a sequence of linear diophantine equations which states an asymptotic property of the sets of their non-negative solutions. This theorem may deserve attention for its own sake.

Let A be a finite set of addresses, A^* the free monoid over A and N_0 the set of natural numbers including zero.

Definition 1.1: A *reference-structure* is a total recursive function $R: N_0 \to A^*$. $R(N_0)$ is the set of address-reference sequences.

The object of this paper will be the second of the following two examples:

Example 1: Let $A=\{a_1, \ldots, a_b\}$. $R: N_0 \to A^*$ given by $R(n)=(a_1 \ldots a_b)^n$ is called the *simple loop over A*.

Example 2: Let $A=\bigcup_{i=1}^{n} A_i, A_i \cap A_j=\emptyset$ for $i \neq j$, $A_i=\{a_1^{(i)}, \ldots, a_{b_i}^{(i)}\}$, $w_i=a_1^{(i)} \ldots a_{b_i}^{(i)}$. Let $\phi: N_0 \to N_0^n$ be a bijective total recursive function, $\phi(m)=(\phi_1(m), \ldots, \phi_n(m))$, $m \in N_0$. Then $R: N_0 \to A^*$ given by $R(m)=w_1^{\phi_1(m)} \ldots w_n^{\phi_n(m)}$ is called the $(A_1, \ldots, A_n)$-*loop-chain.* Obviously, $R(N_0)=\{w_1^{l_1} \ldots w_n^{l_n} \mid l_i \geq 0, 1 \leq i \leq n\}$.

Now let X be a finite set (of "pages") of sufficiently large cardinality, let $p \in N$ be fixed, and A as in 1.1.

Definition 1.2: A $(p-)$ *pagination* of A is a total map $h: A \to X$ such that for all $x \in X: |h^{-1}(x)| \leq p$.

h will also denote the natural extension $h^*: A^* \to X^*$. We wish to process strings in X, i.e. page-references, by paging-algorithms which will be defined as in [6]. Let $m \in N$ be given.

Definition 1.3: An $(m-)$ *paging-automaton* is a 5-tuple $P=(Q_P, q_{0P}, X, \delta_P, \tau_P)$ where: $-Q_P$ is a finite or countably infinite set of states, $-q_{0P} \in Q_P$ is a distinguished initial state, $-X$ serves as set of input symbols, $-\delta_P: Q_P \times X \to Q_P$ and $\tau_P: Q_P \to 2^X$ are total mappings such that for all $x \in X$ and $q \in Q_P$:

(i) $x \in \tau_P(\delta_P(q, x))$,
(ii) $\tau_P(q_{0P})=\emptyset$,
(iii) $|\tau_P(q)| \leq m$, there exists $q \in Q_P$ such that $|\tau_P(q)|=m$,
(iv) $\tau_P(\delta_P(q, x)) \subseteq \tau_P(q) \cup \{x\}$.

P is finite iff Q_P is finite.

We denote by $\mathbf{P}_m$ the class of all finite m-paging-automata. P will be omitted as a subscript whenever no confusion arise.

If $P \in \mathbf{P}_m$ then $k: Q \times X \to N_0$ defined by $k(q, x) = |\tau\,\delta(q, x) - \tau(q)|$ is the *cost-function of P*.

Let δ also denote the sequential extension $\delta^*: Q \times X^* \to Q$ of δ. Then k can be naturally extended to $K: Q \times X^* \to N_0$ by:

$$K(q, \square) = 0,\ q \in Q\ (\square = \text{empty word in } X^*)$$

$$K(q, w\,x) = K(q, w) + k(\delta(q, w), x),\ q \in Q,\ x \in X,\ w \in X^*.$$

We put $K(w) := K(q_0, w)$, $w \in X$. Via K (the "frequency" of page-faults) we shall now define a cost-function on 2^{X^*}: Let $L \subseteq X^*$, $L_n = L \cap X^n$. Then

$$M_P(L_n) = \begin{cases} \frac{1}{|L_n|} \sum K_P(w) & \text{if } L_n \neq \emptyset \\ 0, & \text{else} \end{cases}$$

and

$$C_P(L) = \begin{cases} \max\{\frac{1}{n} M_P(L_n) \mid n > 0\} & \text{if } L \text{ is finite} \\ \limsup\limits_{n \to \infty} \frac{1}{n} M_P(L_n), & \text{else} \end{cases}$$

$C_P(L)$ may be interpreted as an asymptotic mean page-fault-rate.

Definition 1.4: Let $\mathbf{P} \subseteq \mathbf{P}_m$, $\mathbf{H}$ be a set of paginations, and R be a reference-structure. $\mathbf{C}_{\mathbf{P},\mathbf{H}}(R) = \inf\{C_P(h(R(N))) \mid P \in \mathbf{P},\ h \in \mathbf{H}\}$ is called the $(\mathbf{P}, \mathbf{H})$-*paging-complexity* of R.

In [6] the following result has been shown:

Theorem 1.5: *Let R be the simple loop over A. Then there exists* $P_0 \in \mathbf{P}_m$ *and a p-pagination* h_0 *such that for all* $n > 0$:

$K_{P_0}(h_0(R(n))) \le K_P(h(R(n)))$ *for any other pair* (P, h) *and* h_0 *is unique up to permutation of images.*

As expected h_0 reflects the obvious approach of writing adjacent addresses on the same page if possible. (This is no longer true for arbitrary P as has also been shown in [6]!) The following corollaries are readily obtained:

Corollary 1.6: *Let R be the simple loop over A. Then* $C_{P_0}(h_0(R(N))) = \mathbf{C}_{\mathbf{P}_m,\mathbf{H}}(R)$, *where* $\mathbf{H}$ *is the set of all p-paginations.*

Hence (P_0, h_0) is an "optimal pair" with respect to R.

Corollary 1.7: *Let R be the simple loop over A. Then*

$$\mathbf{C}_{\mathbf{P}_m,\mathbf{H}}(R) = \frac{1}{b} \max\{0, k-m\} \left(1 + \frac{1}{k-1}\right),\ \textit{where}\ k = \left\lceil \frac{|A|}{p} \right\rceil.$$

Here, as in the sequel, $\lceil\ \rceil$ denotes the "ceiling"-function on the reals. Corollary 1.7 is a consequence of Corollary 1.6 and the following lemma which can be easily verified by direct calculation:

Lemma 1.8: *Let R be the simple loop over A and let $v_n = h_0(R(n))$, $n \in N$. Then $K_{P_0}(v_n) = k + (uk+t)(k-m) + t - t'$, where $u = (n-1-t)/(k-1) \in N$ for suitable t, $0 \leq t < k-1$, and $t' = \min\{t, m-1\}$; $k = \left\lceil \frac{|A|}{p} \right\rceil$.*

In the present paper we want to show the following theorem:

Theorem 1.9: *Let R be the $(A_1, \ldots, A_n)$-loop-chain, let R_i be the simple loop over A_i, $1 \leq i \leq n$. Then*

$$\mathbf{C}_{\mathbf{P}_m, \mathbf{H}}(R) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{C}_{\mathbf{P}_m, \mathbf{H}}(R_i),$$

where $\mathbf{H}$ is the set of all p-paginations.

The proof of Theorem 1.9 will be postponed to section 3.

2. A Theorem on Linear Diophantine Equations

Definition 2.1: Let $f, g\colon N \to \mathbf{R}$ be sequences of real numbers. f and g are said to be asymptotically equal (in symbols: $f \sim g$ or $f(i) \sim g(i)$) iff $\lim_{i\to\infty} f(i)\, g(i)^{-1}$ exists and $\lim_{i\to\infty} f(i)\, g(i)^{-1} = 1$.

Now consider the linear diophantine equation

$$(\mathbf{X}) \qquad b_1 x_1 + b_2 x_2 + \ldots + b_n x_n = l,$$

where $b_i, l \in N$, $i = 1, \ldots, n$; $n > 1$ and $gcd(b_1, \ldots, b_n) = 1$. The b_i, $1 \leq i \leq n$, will be kept fixed in the sequel. We put:

$\mathbf{D}(l, n) = \{d = (z_1, \ldots, z_n) \mid \sum_{i=1}^{n} b_i z_i = l,\ z_i \geq 0\}$. It is an elementary result that $\mathbf{D}(l, n) \neq \emptyset$ for all sufficiently large l.

With $D(l, n) = |\mathbf{D}(l, n)|$ and the projections $\pi_i(d) = \pi_i(z_1, \ldots, z_n) = z_i$ for $i = 1, \ldots, n$ we define:

$$Z_i(l) = \begin{cases} 0, & \text{if } D(l, n) = 0 \\ \dfrac{b_i}{D(l, n)} \displaystyle\sum_{d \in \mathbf{D}(l, n)} \pi_i(d), & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n$. Observe that $Z_i(l)$ is the arithmetic mean of all multiples of b_i occuring in type $(\mathbf{X})$ partitions of l.

Theorem 2.2: *For all $i, j \in \{1, \ldots, n\}$: $Z_i \sim Z_j$.*

We prove Theorem 2.2 by means of generating functions (cf. [2], for example).

Lemma 2.3: *The generating function of the $D(l, n)$, $l = 0, 1, 2, \ldots$, is*

$$F(z) = \sum D(l, n)\, z^l = ((1 - z^{b_1}) \ldots (1 - z^{b_n}))^{-1}.$$

The proof of Theorem 2.2 also rests on the following asymptotic estimate of $D(l, n)$ which is due to Laguerre. We quote [4], I. 27:

Lemma 2.4:

$$D(l,n) \sim \frac{l^{n-1}}{b_1 \dots b_n (n-1)!}$$

We still need two technical lemmas:

Lemma 2.5: *Let $f, f', g, g': N \to \mathbf{R}_+$ be sequences of positive reals; let $S_f(k) = \sum_{i=1}^{k} f(i)$, $S_{f'}$ etc. correspondingly. Suppose:*

(i) $f \sim f', g \sim g'$;

(ii) $\lim_{k\to\infty} S_f(k) = \lim_{k\to\infty} S_g(k) = \infty$.

Then $\lim S_f(k) S_g(k)^{-1} = \lim S_{f'}(k) S_{g'}(k)^{-1}$ *in the following sense: if one limit exists then the other limit exists and both limits are equal.*

Proof: $f \sim f'$ means $f = f' + \phi$ where $\phi = o(f')$ (i.e. $\phi/f' \to 0$). Hence $S_f = S_{f'} + \Phi$, where $\Phi(k) = \sum_{i=1}^{k} \phi(i)$. We show $\Phi = o(S_{f'})$.

Let $\varepsilon > 0$ be given. By assumption there exists $i_0 \in N$ such that

$$i > i_0 \Rightarrow |\phi(i)| \; |f'(i)|^{-1} \le \varepsilon/2,$$

so: $i > i_0 \Rightarrow |\phi(i)| \le (\varepsilon/2)\, |f'(i)| = (\varepsilon/2) f'(i)$. Let $K = |\phi(1)| + \dots + |\phi(i_0)|$.

Choose $k_0 \in N$ such that $k_0 > i_0$ and $K \left(\sum_{i=1}^{k_0} f'(i) \right)^{-1} \le \varepsilon/2$. Now let $k > k_0$. Then:

$$\left| \frac{\Phi(k)}{S_{f'}(k)} \right| = \frac{\left| \sum_{i=1}^{k} \phi(i) \right|}{\sum_{i=1}^{k} f'(i)} \le \frac{\sum_{i=1}^{k} |\phi(i)|}{\sum_{i=1}^{k} f'(i)} = \frac{\sum_{i=1}^{i_0} |\phi(i)| + \sum_{i=i_0+1}^{k} |\phi(i)|}{\sum_{i=1}^{k} f'(i)}$$

$$\le \frac{K + (\varepsilon/2) \sum_{i=i_0+1}^{k} f'(i)}{\sum_{i=1}^{k} f'(i)} \le \frac{K}{\sum_{i=1}^{k} f'(i)} + \frac{\varepsilon}{2} \le \varepsilon.$$

In like manner one has $S_g = S_{g'} + o(S_{g'})$. The claim of the lemma follows. □

Lemma 2.6: *Let $f: N \to \mathbf{R}$ be a sequence of reals such that $c = \lim_{i\to\infty} f(k + im)$ for some fixed $m > 1$ and all k, $0 \le k < m$. Then $\lim_{i=\infty} f(i)$ exists and $c = \lim_{i=\infty} f(i)$.*

Proof: Obvious.

Proof of Theorem 2.2: Only the case $Z_i(l) \neq 0$ is of interest. We rewrite $Z_i(l)$ as follows:

$$Z_i(l) = \frac{b_i}{D(l,n)} \sum_{d \in \mathbf{D}(l,n)} \pi_i(d) = \frac{b_i}{D(l,n)} \sum_{\substack{x_1, \dots, x_n \ge 0 \\ \sum b_j x_j = l}} x_i.$$

Now for any λ, $1 \le \lambda \le \left\lfloor \frac{l}{b_i} \right\rfloor$ ($\lfloor\ \rfloor$ is the "floor" function!), the number of x_i's such that $x_i = \pi_i(d) = \lambda$ equals the number of nonnegative solutions of

$$b_1 x_1 + \ldots + b_{i-1} x_{i-1} + b_{i+1} x_{i+1} + \ldots + b_n x_n = l - \lambda b_i.$$

Denote this latter number by $D_i(l-\lambda b_i, n-1)$. Hence we can write:

$$Z_i(l) = \frac{b_i}{D(l,n)} \sum_{\lambda=1}^{\lfloor \frac{l}{b_i} \rfloor} \lambda D_i(l-\lambda b_i, n-1).$$

By Lemma 2.3 the generating function of the $D(l,n)$'s is $F(z) = \big((1-z^{b_1}) \ldots \ldots (1-z^{b_n})\big)^{-1}$. Correspondingly we obtain the generating function of the $D_i(l, n-1)$'s: $G_i(z) = F(z)(1-z^{b_i})$, $i = 1, \ldots, n$. We compute $D_i(l-\lambda b_i, n-1)$. First we get:

$$D_i(m, n-1) = \frac{1}{m!} \left[\frac{d^m G_i}{dz^m}\right]_0$$

($[\;]_0$ denotes the value of the m-th derivative at $z=0$). Let $m \geq b_i$; then:

$$\frac{1}{m!} \left[\frac{d^m G_i}{dz^m}\right]_0 = \frac{1}{m!} \left[\frac{d^m F}{dz^m}\right]_0 - \frac{1}{(m-b_i)!} \left[\frac{d^{m-b_i} F}{dz^{m-b_i}}\right]_0.$$

Thus, for $m = l - \lambda b_i$, $\lambda = 1, \ldots, \left\lfloor \frac{l}{b_i} \right\rfloor$, we obtain:

$$D_i(l-\lambda b_i, n-1) = \begin{cases} D(l-\lambda b_i, n) - D\big(l-(\lambda+1) b_i, n\big), & \text{for } 1 \leq \lambda < \left\lfloor \frac{l}{b_i} \right\rfloor \\ D(l-\lambda b_i, n), & \text{for } \lambda = \left\lfloor \frac{l}{b_i} \right\rfloor, \end{cases}$$

which implies: $\sum_{\lambda=1}^{\lfloor \frac{l}{b_i} \rfloor} \lambda D_i(l-\lambda b_i, n-1) = \sum_{\lambda=1}^{\lfloor \frac{l}{b_i} \rfloor} D(l-\lambda b_i, n)$, and hence:

$$Z_i(l) = \frac{b_i}{D(l,n)} \sum_{\lambda=1}^{\lfloor \frac{l}{b_i} \rfloor} D(l-\lambda b_i, n).$$

Now consider $Z_i(l)$ and $Z_j(l)$, $i \neq j$. We put $m = lcm(b_i, b_j) = \frac{b_i b_j}{gcd(b_i, b_j)}$. With $l = pm + r$ for some $p \geq 0$ and r, $0 \leq r < m$, we get:

$$\left\lfloor \frac{l}{b_i} \right\rfloor = p \frac{b_j}{gcd(b_i, b_j)} + \left\lfloor \frac{r}{b_i} \right\rfloor =: u_i(p, r)$$

$$\left\lfloor \frac{l}{b_j} \right\rfloor = p \frac{b_i}{gcd(b_i, b_j)} + \left\lfloor \frac{r}{b_j} \right\rfloor =: u_j(p, r).$$

This implies:

$$\left\{l - \lambda b_i \mid \lambda = 1, \ldots, \left\lfloor \frac{l}{b_i} \right\rfloor\right\} = \left\{r_i + \lambda b_i \mid \lambda = 0, \ldots, u_i(p, r);\ r_i = r - b_i \left\lfloor \frac{r}{b_i} \right\rfloor\right\}$$

and correspondingly for b_j. We obtain:

$$\frac{Z_i(l)}{Z_j(l)}=\frac{Z_i(r+pm)}{Z_j(r+pm)}=:\frac{Z_i(p,r)}{Z_j(p,r)}=\frac{b_i}{b_j}\;\frac{\sum\limits_{\lambda=0}^{u_i(p,r)-1} D(r_i+\lambda b_i, n)}{\sum\limits_{\lambda=0}^{u_j(p,r)-1} D(r_j+\lambda b_j, n)}$$

By Lemma 2.4: $D(l,n)\sim\dfrac{l^{n-1}}{b_1\ldots b_n(n-1)!}$. Hence, by an immediate slight modification of Lemma 2.5, taking into account the different but — as p increases — equally growing upper summation bounds in the above sums, we get:

$$\lim_{p\to\infty}\frac{Z_i(p,r)}{Z_j(p,r)}=\frac{b_i}{b_j}\lim_{p\to\infty}\frac{\sum\limits_{\lambda=0}^{u_i(p,r)-1}(r_i+\lambda b_i)^{n-1}}{\sum\limits_{\lambda=0}^{u_j(p,r)-1}(r_j+\lambda b_j)^{n-1}}.$$

By the elementary expansions

$$\sum_{\lambda=0}^{u-1}(r+\lambda b)^{n-1}=\sum_{\mu=0}^{n-1}\binom{n-1}{\mu}r^{n-\mu-1}b^{\mu}\sum_{\lambda=1}^{u-1}\lambda^{\mu}$$

and

$$\sum_{\lambda=1}^{u-1}\lambda^{n-1}=\frac{(u-1)^n}{n}+O\left((u-1)^{n-1}\right),$$

the term of highest order in

$$\sum_{\lambda=0}^{u-1}(r+\lambda b)^{n-1} \text{ is } b^{n-1}\frac{(u-1)^n}{n}.$$

So the limit becomes:

$$\lim_{p\to\infty}\frac{Z_i(p,r)}{Z_j(p,r)}=\lim_{p\to\infty}\frac{b_i^n(u_i(p,r)-1)^n+O\left(u_i(p,r)^{n-1}\right)}{b_j^n(u_j(p,r)-1)^n+O\left(u_j(p,r)^{n-1}\right)}$$

$$=\lim_{p\to\infty}\frac{b_i^n\left(p\dfrac{b_j}{gcd(b_i,b_j)}+\left\lfloor\dfrac{r}{b_i}\right\rfloor-1\right)^n}{b_j^n\left(p\dfrac{b_i}{gcd(b_i,b_j)}+\left\lfloor\dfrac{r}{b_j}\right\rfloor-1\right)^n}=1$$

The claim now follows via Lemma 2.6. □

The following corollaries are immediate:

Corollary 2.7: *For all* $j=1,\ldots,n$: $Z_j(l)\sim\dfrac{l}{n}$.

Corollary 2.8: *For all* $j=1,\ldots,n$: $\lim\limits_{l\to\infty}\dfrac{Z_j(l)}{l}=\dfrac{1}{n}$.

Corollary 2.9: *Let* $\sum\limits_{i=1}^{n} b_i x_i=l$, $l=1,2,\ldots$, *be a sequence of linear diophantine equations like* **(X)** *but with* $gcd(b_1,\ldots,b_n)>1$. *Then for all* $j=1,\ldots,n$: $\limsup\limits_{l\to\infty}\dfrac{Z_j(l)}{l}=\dfrac{1}{n}$.

3. The Paging-Complexity of Loop-Chains

We are now ready to prove Theorem 1.9. For notational convenience we put for $P \in \mathbf{P}_m$ and $h \in \mathbf{H}$: $K_{P,h}(w) := K_P(h(w))$, $C_{P,h}(R) := C_P(h(R(N)))$, where $w \in A^*$ and R is a reference-structure. Now let R be an $(A_1, \ldots, A_n)$-loop-chain. The statement of Theorem 1.9 is trivial if $n=1$. Hence let $n>1$, let $P \in \mathbf{P}_m$ and $h \in \mathbf{H}$. The theorem follows if we can show:

Lemma 3.1: $C_{P,h}(R) \geq \frac{1}{n} \sum_{i=1}^{n} \mathbf{C}_{\mathbf{P}_m, \mathbf{H}}(R_i)$.

Lemma 3.2: *The estimate of Lemma* 3.1 *is sharp.*

Proof of 3.1: First observe that Theorem 1.5 and Lemma 1.8 imply: For all $w \in A^*$, $y \in N$, $y>1$:

$$K_{P,h}(w w_i^y) - K_{P,h}(w w_i) \geq \frac{y-(t+1)}{k_i - 1} k_i (k_i - m) + t(k_i - m) + t - t'$$

$$= y \frac{k_i}{k_i - 1}(k_i - m) + \left(t - t \frac{k_i}{k_i - 1} - \frac{k_i}{k_i - 1}\right)(k_i - m) + t - t',$$

if $k_i := \left\lceil \frac{b_i}{p} \right\rceil > m$, and $K_{P,h}(w w_i^y) - K_{P,h}(w w_i) \geq 0$, otherwise. Here, as in Lemma 1.8: $y-1 = u(k_i - 1) + t$ with $0 \leq t < k_i - 1$ for suitable $u \in N$, $t' = \min\{t, m-1\}$; $w_i = a_1^{(i)} \ldots a_{b_i}^{(i)}$ (cf. Example 2!).

Now $\left|\left(t - t\frac{k_i}{k_i - 1} - \frac{k_i}{k_i - 1}\right)(k_i - m) + t - t'\right|$ is bounded, and hence

$$K_{P,h}(w w_i^y) - K_{P,h}(w w_i) \geq \begin{cases} y \frac{k_i}{k_i - 1}(k_i - m) + c_i, & \text{if } k_i > m \\ 0, & \text{otherwise,} \end{cases}$$

where the c_i, $1 \leq i \leq n$, are suitable constants, independent of y. Now let $w \in R(N_0)$, i.e. $w = w_1^{y_1} \ldots w_n^{y_n}$, $y_i \in N_0$, $i = 1, \ldots, n$. By the preceding estimate we get:

$$K_{P,h}(w) \geq \sum \left(y_i \frac{k_i}{k_i - 1}(k_i - m) + c_i\right),$$

where the sum is over all i such that $k_i > m$.

We compute $M_{P,h}(R, l) := M_P((h(R(N_0)))_l)$: For w as above we have:

$L(w) = \sum_{i=1}^{n} b_i y_i$ (where $L(\cdot)$ denotes the length-function on A^*). So $L(w) = l$ if and only if $\sum_{i=1}^{n} b_i y_i = l$, and $M_{P,h}(R, l) = 0$ if and only if $D(l, n) = 0$, with $D(l, n)$ as defined in section 2. Now let $D(l, n) \neq 0$. Then

$$M_{P,h}(R, l) = \frac{1}{D(l, n)} \sum_{\substack{w \in R(N_0) \\ L(w) = l}} K_{P,h}(w)$$

$$\geq \frac{1}{D(l,n)} \sum_{\substack{y_1,\ldots,y_n \geq 0 \\ \sum b_j y_j = l}} \sum_{i:k_i>m} \left(y_i \frac{k_i}{k_i-1} (k_i-m)+c_i \right)$$

$$= \sum_{i:k_i>m} \left(\frac{1}{D(l,n)} \sum_{\substack{y_1,\ldots,y_n \geq 0 \\ \sum b_j y_j = l}} b_i y_i \right) \frac{1}{b_i} \frac{k_i}{k_i-1} (k_i-m)+C$$

where $C = \sum_{i:k_i>m} c_i$ is a constant independent of l. By Corollary 1.7 we have:

$$\mathbf{C}_{\mathbf{P}_m,\mathbf{H}}(R_i) = \begin{cases} \frac{1}{b_i} \frac{k_i}{k_i-1} (k_i-m), & \text{if } k_i > m \\ 0, & \text{otherwise,} \end{cases}$$

Hence, employing the functions Z_i as defined in section 2, we can write:

$$M_{P,h}(R,l) \geq \sum_{i=1}^{n} Z_i(l)\, \mathbf{C}_{\mathbf{P}_m,\mathbf{H}}(R_i)+C,$$

whence, by Corollary 2.9 and the definitions of section 1:

$$C_{P,h}(R) = \limsup_{l\to\infty} \frac{1}{l} M_{P,h}(R,l) \geq \limsup_{l\to\infty} \sum_{i=1}^{n} \frac{Z_i(l)}{l} \mathbf{C}_{\mathbf{P}_m,\mathbf{H}}(R_i)$$

$$= \sum_{i=1}^{n} \frac{1}{n} \mathbf{C}_{\mathbf{P}_m,\mathbf{H}}(R_i), \text{ the desired result.} \quad \square$$

Proof of 3.2: The lower bound in Lemma 3.1 is assumed by any pair (P_0, h_0), where — for each A_i, $i=1, \ldots, n$, separately — h_0 is as described after Theorem 1.5, and P_0 realizes an optimal strategy (cf. [1]) on simple loops, cf. for instance [6]. □

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